for a publisher like Springer, and particularly lugubrious for a book commanding such a high price tag. Moreover, some of the transliterations are awkward; Lyapunov and Bunyakovski are more common than Liapounoff and Bouniakovsky.

Nonetheless, this book is destined to become a classic. It should be read by every historian of mathematics, and should be a part of one's personal library, especially if it can be purchased at a reduced price.

REFERENCES

Boyer, C. B., & Merzbach, U. 1989. *A history of mathematics*. New York: Wiley.

Eves, H. 1990. *An Introduction to the History of Mathematics*. Philadelphia: Saunders.

Ewing, J. 1992. Essay review: The history of mathematics. *Historia Mathematica* **19,** 93–98.

Kiernan, B. M. 1971. Galois theory from Lagrange to Artin. *Archive for History of Exact Sciences* **8,** 40–154.

Kline, M. 1972. *Mathematical Thought from Ancient to Modern Times*. New York: Oxford Univ. Press.

Neuenschwander, E. 1984. Joseph Liouville (1809–1882): Correspondance inédite et documents biographiques provenant de différentes archives parisiennes. *Bollettino di Storia delle Scienze Matematiche* **4**(2), 55–132.

——— 1990. The unpublished papers of Joseph Liouville in Bordeaux. *Historia Mathematica* **16,** 334–342.

Struik, D. J. 1987. *A Concise History of Mathematics*. New York: Dover.

Taton, R. 1973. Liouville, Joseph. In *Dictionary of Scientific Biography*, C. C. Gillispie, Ed., Vol. 8, pp. 381–387. New York: Scribner's.

The Space of Mathematics. Philosophical, Epistemological, and Historical Explorations. Edited by Javier Echeverria, Andoni Ibarra, and Thomas Mormann. Berlin, New York (de Gruyter). 1992. xvi + 422 pp. With an index of names.

REVIEWED BY MORITZ EPPLE

Fachbereich Mathematik, Universität Mainz, Postfach 3980, D-W-6500 Mainz, Germany

The historiography of mathematics has a number of open boundaries, such as those running toward contemporary mathematics, or toward the general historiography of the natural sciences. Not all of these boundary lines or regions are associated with unproblematical transverse discourses in which ideas and concepts are exchanged between the adjoining sides. One of these critical boundaries is that separating the historiography and the philosophy of mathematics. The twentieth century has witnessed the professionalization of both of these fields, together with the stabilization of their respective discursive practices. Still, for some time the concern for internal coherence has limited the themes for mutual discussion between historians and philosophers to a rather narrow range, including, for example, the emergence of set theory, Hilbert's metamathematical program, or Gödel's results.

In recent years, however, the situation seems to have changed. Congresses on the philosophy of mathematics include historical sessions, and historical discussions acknowledge the importance of adequate conceptual frameworks for the understanding of the development of mathematics. The book under review here is one of the publications documenting this mutual interest in the history and philosophy of mathematics [1]. At the same time, it shows the difficulties of exploring a region of discourse which is not yet structured by a set of generally accepted concepts, problems, and fundamental beliefs.

The volume contains revised versions of papers presented at a multidisciplinary symposium on "Structures in Mathematical Theories" which took place in San Sebastian, Spain, in 1990. The disciplines represented at the symposium ranged from the history of mathematics (contributors are, among others, Joseph W. Dauben, Ivor Grattan-Guinness, Eberhard Knobloch, and Michael S. Mahoney) and philosophy of mathematics (Solomon Feferman, Michael D. Resnik, *et al.*) to the philosophy of science (Joseph D. Sneed, Erhard Scheibe, *et al.*) and mathematics itself (Saunders MacLane, F. William Lawvere).

In their "introductory afterthoughts," the editors characterize this volume as part of a "rather dramatic transformation and reorientation" (p. ix) in the field of the philosophy of mathematics. This transformation process is supposed to represent a shift from what the editors call the "Neo-Fregean approach," focusing on the problem of logical foundations of mathematics, toward a "new" philosophy of mathematics, emphasizing empiricist conceptions of what mathematics is about and how mathematics develops [2]. Connected with this empiricist attitude is the conviction that it is misleading to consider only arithmetic or set theory as the domains of mathematics to be explained philosophically [3]. Instead, philosophy of mathematics should take the whole range of mathematical activity into account and, in particular, its roots in and applications to the "real world." Accordingly, many contributions to the volume discuss from one side or another problems regarding the applicability of mathematics, the relation between mathematics and physics, or the "quasiempirical" methodology of different branches of mathematics.

As the introduction suggests, the emphasis of the editors is on the *philosophy* of mathematics, and historical contributions are mainly thought of as providing evidence for the "new" empiricist position. In fact, a similar view can be found in some of the articles. Of the twenty-three papers collected in the volume, only six are explicitly intended as contributions to the history of mathematics. Whether the purely philosophical papers in fact contain arguments in favor of the "new" trend in the philosophy of mathematics will not be discussed here. Rather, the "multidisciplinary" intentions of the editors will be taken seriously. Now, a multidisciplinary perspective (or, rather, multiple disciplinary perspectives) can contribute to discourse across the common boundary lines only when one perspective has something to offer which appeals to the professional view of the other discipline(s). A reader interested in the open space between the history and philosophy of mathematics will thus probably ask her- or himself which of the historical articles collected in the volume offer new or challenging information for philosophers and, similarly, whether some of the philosophical contributions may serve to stimulate historical discussions. In fact, there appear to be some potentially interesting topics gathered together in this book, and the remainder of this review will concentrate on them.

From history to philosophy. A nice historical case study in favor of the empiricist trend in the philosophy of mathematics is Javier Echeverria's "Observations, Problems, and Conjectures in Number Theory—The History of the Prime Number Theorem" (pp. 230–252). Echeverria's aim is to confirm the Polya–Lakatos account of mathematical practice as a process of inductive reasoning, guided by experience and intuition, for certain key developments in the field of number theory. Drawing on classical historical studies (e.g., [Landau 1909, Dickson 1919]), he recounts the history of the conjectures on the distribution of primes from Euler's queries to the first proofs of the prime number theorem shortly before the turn of the twentieth century, with an emphasis on the history of prime number tables as testing devices for the different conjectures. Particular attention is given to the work of Gauss and Legendre, who extensively used and even improved the prime number tables of their time. Echeverria states his main thesis as follows: "Prime number tables play the same epistemological role in number theory as empirical measurements and data in experimental science" (p. 250). While it will not come as a surprise for historians that number theory has a rather empirical character, it may indeed be an interesting question for an epistemology of mathematics to account for such empirical aspects of the process of gaining mathematical knowledge. Echeverria suggests that this experimental acquisition of knowledge is not incidental, but typical in situations where a mathematical problem stands open for a long period of time. One might even ask whether Echeverria does not understate the phenomenon in question by reducing the empirical aspect of mathematical research to empirical tests of conjectures. Can tentative steps toward the proof of partial or related results, attempts to establish connections with previous knowledge, and similar elements of mathematical practice not also be described as an experimental process?

Two other papers dealing with nonstandard phenomena in the history of mathematics are Joseph W. Dauben's "Are There Revolutions in Mathematics?" (pp. 205–229) and Ivor Grattan-Guiness' "Structure-Similarity as a Cornerstone of the Philosophy of Mathematics" (pp. 91–111). Dauben's article makes clear that the main problem in speaking of "revolutions" in the history of mathematics lies in the specific *notion* of a revolution which one chooses to use. While Dauben stresses several diachronical breaks in mathematics, Grattan-Guinness points to certain synchronical analogies and interconnections between mathematical and physical theories, which he calls "structure-similarities." An example would be the use of the notion of linearity in different branches of nineteenth-century mathematics and physics. Again, a precise *notion* of such analogies would be very interesting philosophically. However, both texts leave the necessary conceptual clarifications to the reader.

It seems that such a clarification could start from semantic considerations. The key notions of both papers depend on an explanation of the *meaning* of mathematical concepts. "Revolutions" are at least partially to be thought of as breaks in the semantics of mathematical language (Dauben, p. 208; cf. also Kitcher's thoughts on the "modification of reference potentials" [Kitcher 1983, 170]), and "structure-similarities" are intended as explanations of "how a mathematical theory can mean" (Grattan-Guinness, p. 93). Neither of the authors is prepared, however, to go into the discussions of the semantics of mathematical language which have been central to the philosophy of mathematics since its emergence from the foundational debates at the beginning of our century.

Perhaps the most interesting challenge for the philosophy of mathematics is the historical contribution of Michael S. Mahoney, "Computers and Mathematics: The Search for a Discipline of Computer Science" (pp. 349–363). Mahoney reviews the emergence of the mathematical theory of computing from Turing and Shannon to the present day, stressing the fragile status of the theoretical conceptions linking real calculating machines on the one hand and mathematical logics or formal semantics on the other. In computer science, nothing like a synthesis of a mathematical theory with a range of practical applications, comparable to that of Newtonian mechanics, is yet available. At the end of his presentation, Mahoney suggests that this is perhaps not due to the youth of computer science, but may rather be seen as a sign that a different constellation of relationships exists between mathematics and its applications in the realm of computing. While in the case of classical applications of mathematics, "mathematization had elicited the essential simplicity of an apparently complex world," the mathematization of computing has probably to acknowledge the irreducible complexity of computations (p. 362f). Thus, Mahoney concludes, "the search for a mathematical structure of computing may well involve a new historical and philosophical structure of mathematization" (p. 363).

From philosophy to history. An interesting debate in the recent philosophy of mathematics centers on the so-called "indispensability argument." The argument, which goes back to W. V. O. Quine and H. Putnam, is concerned with the reality of mathematical objects. Mathematical statements are indispensable in doing science, so it is argued, and therefore their referents (for example, quaternions) should be considered exactly as real as those of, say, physical statements (for instance, atoms). Michael D. Resnik, in his article "Applying Mathematics and the Indispensability Argument" (pp. 115–131) tries to defend this argument against recent criticisms; for example, those of Hartry Field, who has claimed in his book *Science Without Numbers* [Field 1980] that the fruitful application of mathematics in science does not depend on its being true or on the existence of its objects.

From an *historical* point of view, the interesting point of the argument, possibly counter to the intention of its defenders, is that the dispensability or indispensability of mathematical concepts seems to vary historically. Newton thought the mathematics of "angles of contact" between curved arcs and their tangents was indispensable in his presentation of the motion of bodies; modern textbooks do not. How indispensable was mathematics at different stages of the history of science? Exactly which mathematical objects were indispensable? Does the argument lead to an historically changing ontology of mathematics [4]? Does the indispensability of mathematics for the empirical sciences bring about the epistemic or social legitimation of mathematics? These questions would seem to open an interesting field of historical inquiry to which philosophers will hardly address themselves.

Not quite in the mainstream of the traditional epistemology of mathematics lies the distinction between "explicit" and "tacit" knowledge, due to Michael Polanyi [1958], on which Herbert Breger focuses in his "Tacit Knowledge in Mathematical Theory" (pp. 79–90). Breger offers a typology of tacit knowledge, together with some examples illustrating it. He conceives tacit knowledge mainly as a form of "know-how" [5] and distinguishes know-how for problem-solving, know-how for finding the right definitions and generalizations, and know-how for adequate axiomatization. Moreover, he lists the general "understanding of a theory" as well as the "knowledge of the trivial" (that which one cannot find in mathematical texts) as forms of tacit knowledge. While it is not clear whether these distinctions are more than an ad hoc typology, Breger is probably right in his opinion that understanding the role of tacit knowledge in mathematics provides a clue to some methodological questions in the history of mathematics. It is clear that the tacit knowledge of a mathematical community changes. This can, first, be one of the reasons for the strangeness of old texts, which originally assumed that the reader was acquainted with a tacit knowledge different from ours (p. 80). Second, the process of making explicit earlier tacit knowledge certainly represents a rather general pattern of mathematical change (p. 82; Breger's example is the axiomatization of algebraic topology). It would be a difficult, but interesting, historical task to reconstruct in some cases the horizon of tacit knowledge which provided, so to speak, the frame of reference for the meaning of the mathematical texts written with this knowledge in mind. On this basis, one could then study which parts of this horizon were eventually transformed into explicit knowledge, and which parts were replaced or forgotten.

While the reviewed papers provoke interesting questions for a substantial discourse between the philosophy and the history of modern mathematics, as I have tried to show, most of the answers remain open. One has the impression that the real work is still to be done. Philosophers seldom rely on more than hints at the history of mathematics, and in some cases are not even aware of or interested in the historical implications of their work. Historians, in turn, often seem to have reservations about entering into a discussion of the different arguments which philosophy of mathematics has brought about so far, and about adopting the standards of conceptual clarity characterizing the best of its contributions.

The contributions to the book under review show these difficulties quite clearly.

It seems that both of the communities involved would have to broaden (and deepen) their understanding of the problems raised in the discussions on the other side of the boundary if they are really interested in exploring "the space of mathematics." This applies to issues such as the philosophical problem of the (in)dispensability of mathematics in empirical sciences and the historical experience of diverging forms of mathematization, both of which call into question the received view of a fixed methodology and ontology of mathematics. Perhaps it would be a good idea to hold a conference like the one leading to the present volume and to invite contributions like "The Semantics of Mathematical Language for Historians" or "The History of *** for Philosophers." Unfortunately, no such contributions are published in *The Space of Mathematics*.

A professional mathematician, such as Saunders Mac Lane, has the easiest part to play in the historico-philosophical game. Knowing very well from his own experience that "mathematics is protean" and that no philosophical or historical perspective will ever capture its essence, he may ignore the difficulties of the latter and content himself with the use of a good metaphor, giving his wise advice: "May protean understanding prosper!" (p. 13).

NOTES

1. Others would be [Aspray & Kitcher 1988], [Hersh 1988], and [Kitcher 1983], the latter of whom risks including in his empiricist epistemology of mathematics a "theory of mathematical change." A comparable text from the other side, which would include a philosophy of mathematics from the point of view of a historian, is still missing.

2. The most representative recent text for the first aspect would probably be [Kitcher 1983]. For the second aspect, the editors and several of the contributors (Grattan-Guiness, Ibarra/Mormann, Niiniluoto, Echeverria) refer to the (not quite so "new") position of Imre Lakatos.

3. In fact, this view is justified only on the basis of a reductionist view of mathematics, be it in the form of the original Fregean and Russellian claim that a philosophical understanding of mathematics consists in the reduction of the concepts of "number," "set," etc. to purely logical concepts, or in the more modern form of explaining mathematics by a reduction to set-theoretical axiomatics.

4. In her talk at the 1992 Wittgenstein symposium on the philosophy of mathematics at Kirchberg, Austria, Penelope Maddy tried to dismiss the indispensability argument on this ground.

5. He does not, however, make the obvious connection to Gilbert Ryle's distinction between "knowing that" and "knowing how" [Ryle 1949] or to the subsequent discussion of this distinction in the analytic theory of knowledge.

REFERENCES

Aspray, W., & Kitcher, Ph. (Eds.) 1988. *History and Philosophy of Modern Mathematics*. Minneapolis: Univ. of Minnesota Press.

Dickson, L. E. 1919. *History of the Theory of Numbers*, Vol. 1. Washington: Carnegie Institution.

Field, H. 1980. *Science without Numbers*. Princeton: Princeton Univ. Press.

Hersh, R. (Ed.) 1988. New directions in the philosophy of mathematics. *Synthese* **88,** No. 2.

Kitcher, Ph. 1983. *The Nature of Mathematical Knowledge*. Oxford: Oxford Univ. Press.

Landau, E. 1909. *Handbuch der Lehre von der Verteilung der Primzahlen*, Vol. 1. Leipzig/Berlin: Teubner.

Polanyi, M. 1958. *Personal Knowledge*. Chicago: Univ. of Chicago Press.

Ryle, G. 1949. *The Concept of Mind*. London: Hutchinson.